Subharmonic steps and inertial effects in a system of two coupled overdamped oscillators

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(Received 28 April 1992; revised manuscript received 18 March 1993)

Using the singular perturbation method, we solved analytically the equation of motion for two coupled nonlinear overdamped oscillators. We demonstrated that the appearance of subharmonic steps can be caused by a coupling term, and its possible relation with an inertial effect of the coupled system was also discussed.

PACS number(s): 03.20.+i, 46.10.+z

Recently, the dynamical behavior of a nonlinear system with many interacting degrees of freedom has been given much attention [1]. One of the best examples is the system of many-coupled nonlinear oscillators. The reason is that this coupled system has a close connection with dynamical responses of many realistic physical systems, e.g., sliding charge-density waves (CDW) [2,3], Josephson junctions [4], and the flux creep and flow dynamics in the high-T_c or type-II superconductors [5], etc. Most of these systems can be modeled by one-dimensional chain of balls which are connected by harmonic springs with spring constant k and move in a sinusoidal potential. Usually, for simplicity, the mass of the ball is neglected and in the case of no coupling, the motion of each ball can be described by an overdamped nonlinear harmonic oscillator. It is well known that even this single overdamped nonlinear oscillator can show the so-called mode-locking phenomenon [6] when it suffers from a combination of dc and ac external forces with frequency ω, i.e., the ball will have a tendency to move with an intrinsic frequency ω_0 which is locked at a series of multiple values of the ω ($\omega_0=n\omega$, where n is an integer). This mode locking is called harmonic. However, the subharmonic mode locking (i.e., $\omega_0=p\omega/q$, where p and q are integers) cannot be observed for the single overdamped nonlinear oscillator. This is an obvious example showing the importance of the coupling between neighboring oscillators. What the real effect is of the coupling on mode locking and other dynamical properties is of interest to many researchers. The coupled equation of motion for the coupled overdamped nonlinear oscillators has been solved by different methods, e.g., discrete mapping [7], the mean-field method [8], and numerical integration [9]. In this paper we try to use the singular perturbation method [10] to solve analytically the equation of motion for two coupled nonlinear oscillators, which is the simplest situation including the effect of the coupling. This analytical method has overcome shortcomings existing in all mappings and numerical calculations, e.g., the dependence of the subharmonic mode locking on the selection of initial conditions, and on the form of the map at the top and bottom of the potential well, etc. We found that the coupling has a fundamental effect on the subharmonic mode locking, i.e., as long as the coupling constant k does not equal zero, the width of the subharmonic steps will not become zero. It is well known that the subharmonic mode locking can also be caused by an inertial term in an underdamped oscillator [11]. Therefore, we may say that coupling between overdamped oscillators can produce an "inertial" effect. The subharmonic mode locking in the dynamical response of an overdamped chain of nonlinear oscillators, and other "inertial" effects, e.g., the ringing [12] existing in the transient dynamics of the nonlinear system and the occurrence of chaos [13], etc., may also be interpreted by the coupling between many degrees of freedom in the nonlinear system [14].

We consider only two relaxation oscillators coupled by a harmonic spring with spring constant $\tilde{k}$ and lying in a sinusoidal potential. The coupled system is driven by a time-dependent force $F(t)$. The equation of motion for the coupled system can be written as

$$\begin{aligned}&\frac{1}{\tau}\dot{x}_1+\tilde{k}(x_1-x_2)-\frac{\omega_r^2}{Q}\sin(Qx_1)=F(t)/m\ ,\\&\frac{1}{\tau}\dot{x}_2+\tilde{k}(x_2-x_1)-\frac{\omega_r^2}{Q}\sin(Qx_2)=F(t)/m\ ,\end{aligned}\tag{1}$$

where x_i represents the position of the ith oscillator, τ is the relaxation time, $Q=2\pi/\lambda$ with λ being the period of the sinusoidal potential, m is the effective mass of the oscillators, and ω_r is the restoring frequency. Usually the external force $F(t)$ should have two parts F_0 and $F_1\cos(\omega t)$, corresponding to a constant force and a vibrating force, respectively. Introducing the phase variables $\varphi_i=Qx_i$, Eq. (1) can be rewritten as

$$\begin{aligned}&\dot{\varphi}_1+k(\varphi_1-\varphi_2)+\sin\varphi_1=e+e_1\cos(\omega t)\ ,\\&\dot{\varphi}_2+k(\varphi_2-\varphi_1)+\sin\varphi_2=e+e_1\cos(\omega t)\ ,\end{aligned}\tag{2}$$

where $e=F_0/F_T$, $e_1=F_1/F_T$, and $F_T=m\omega_r^2/Q$; the time t is measured in units of ω_c^{-1}, $\omega_c=\omega_r^2\tau$, and the frequency ω is measured in units of ω_c; $k=\tilde{k}/\omega_r^2$ is a dimensionless

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coupling constant. In general, the coupled nonlinear equations (2) cannot be analytically solved, but in the limit of $k \ll 1$ and $e_1 \ll 1$ (i.e., in the case of weak coupling and strong pinning, and a small amplitude of the vibrating force), Eq. (2) can be analytically solved by the singular perturbation method.

We first expand the phase variable φ_i in power of the small parameter e_1, and then again expand coefficients of the series in powers of the coupling constant k,

$$\begin{aligned} \varphi_1 &= \psi_1^{(0)} + \psi_1^{(1)} e_1 + \cdots , \\ \varphi_2 &= \psi_2^{(0)} + \psi_2^{(1)} e_1 + \cdots , \\ e &= a_0 + \lambda_1 e_1 + \cdots . \end{aligned} \tag{3}$$

Substituting Eq. (3) into Eq. (2) and collecting e_1 terms with the same order, we obtain the following system of equations.

(1) For the e_1^0 term, we have

$$\begin{aligned} \dot{\psi}_1^{(0)} + k(\psi_1^{(0)} - \psi_2^{(0)}) - \sin\psi_1^{(0)} &= a_0 , \\ \dot{\psi}_2^{(0)} + k(\psi_2^{(0)} - \psi_1^{(0)}) - \sin\psi_2^{(0)} &= a_0 . \end{aligned} \tag{4}$$

Now $\psi_1^{(0)}$, $\psi_2^{(0)}$, and a_0 can be again expanded in powers of k, and here, for simplicity, we keep only up to k^1 terms

$$\psi_1^{(0)} = f_0 + k f_1 , \quad \psi_2^{(0)} = g_0 + k g_1 , \quad a_0 = e_0 + \alpha_1 k . \tag{5}$$

Substituting (5) into (4), collecting also terms of the same order in k, we have the following.

(a) The k^0 term:

$$\begin{aligned} \dot{f}_0 - \sin f_0 &= e_0 , \\ \dot{g}_0 - \sin g_0 &= e_0 . \end{aligned} \tag{6}$$

If $e_0 < 1$, Eq. (6) has no rotational solution, i.e., the oscillators are in a static equilibrium state. Because we are only interested in the dynamical response of the coupled oscillator system to an external force, here we will only take $e_0 > 1$. It is easy to find a solution of Eq. (6) in this case [15]

$$\begin{aligned} \dot{f}_0 &= \frac{e_0^2 - 1}{e_0 + \sin(\omega_0 t + \theta_1)} , \\ \dot{g}_0 &= \frac{e_0^2 - 1}{e_0 + \sin(\omega_0 t + \theta_2)} , \end{aligned} \tag{7}$$

where θ_1 and θ_2 are two integral constants, $\omega_0 = (e_0^2 - 1)^{1/2}$.

(b) The k^1 term:

$$\begin{aligned} \dot{f}_1 + (f_0 - g_0) - f_1 \cos f_0 &= \alpha_1 , \\ \dot{g}_1 + (g_0 - f_0) - g_1 \cos g_0 &= \alpha_1 . \end{aligned} \tag{8}$$

The solution of Eq. (8) is

$$\begin{aligned} f_1 &= (e_0 + \sin f_0) \int \frac{\alpha_1 + (g_0 - f_0)}{e_0 + \sin f_0} dt , \\ g_1 &= (e_0 + \sin g_0) \int \frac{\alpha_1 + (f_0 - g_0)}{e_0 + \sin g_0} dt . \end{aligned} \tag{9}$$

(2) For the e_1^1 term, we have

$$\begin{aligned} \dot{\psi}_1^{(1)} + k(\psi_1^{(1)} - \psi_2^{(1)}) - \psi_1^{(1)} \cos\psi_1^{(0)} &= \lambda_1 + \cos(\omega t) , \\ \dot{\psi}_2^{(1)} + k(\psi_2^{(1)} - \psi_1^{(1)}) - \psi_2^{(1)} \cos\psi_2^{(0)} &= \lambda_1 + \cos(\omega t) . \end{aligned} \tag{10}$$

Similarly, $\psi_1^{(1)}$, $\psi_2^{(1)}$, and λ_1 can also be expanded in powers of k, and we keep only up to k^1 terms

$$\psi_1^{(1)} = F_0 + k F_1 , \quad \psi_2^{(1)} = G_0 + k G_1 , \quad \lambda_1 = b_0 + \beta_1 k . \tag{11}$$

Substituting Eq. (11) into Eq. (10), and collecting terms with the same order of k, we have the following.

(a) The k^0 term:

$$\begin{aligned} \dot{F}_0 - F_0 \cos f_0 &= b_0 + \cos(\omega t) , \\ \dot{G}_0 - G_0 \cos g_0 &= b_0 + \cos(\omega t) . \end{aligned} \tag{12}$$

Its integral solution is

$$\begin{aligned} F_0 &= (e_0 + \sin f_0) \int \frac{b_0 + \cos(\omega t)}{e_0 + \sin f_0} dt , \\ G_0 &= (e_0 + \sin g_0) \int \frac{b_0 + \cos(\omega t)}{e_0 + \sin g_0} dt . \end{aligned} \tag{13}$$

(b) The k^1 term:

$$\begin{aligned} \dot{F}_1 - F_1 \cos f_0 &= \beta_1 - F_0 f_1 \sin f_0 - (F_0 - G_0) , \\ \dot{G}_1 - G_1 \cos g_0 &= \beta_1 - G_0 g_1 \sin f_0 - (G_0 - F_0) . \end{aligned} \tag{14}$$

Its integral solution is

$$\begin{aligned} F_1 &= (e_0 + \sin f_0) \int \frac{\beta_1 - F_0 f_1 \sin f_0 - (F_0 - G_0)}{e_0 + \sin f_0} dt , \\ G_1 &= (e_0 + \sin g_0) \int \frac{\beta_1 - G_0 g_1 \sin g_0 - (G_0 - F_0)}{e_0 + \sin g_0} dt . \end{aligned} \tag{15}$$

Now, we have an integral solution of Eq. (2) as follows:

$$\begin{aligned} \varphi_1 &= \psi_1^{(0)} + e_1 \psi_1^{(1)} = f_0 + k f_1 + e_1 F_0 + e_1 k F_1 , \\ \varphi_2 &= \psi_2^{(0)} + e_1 \psi_2^{(1)} = g_0 + k g_1 + e_1 G_0 + e_1 k G_1 . \end{aligned} \tag{16}$$

At the same time, the constant external force e has been expanded as

$$e = a_0 + \lambda_1 e_1 = e_0 + \alpha_1 k + b_0 e_1 + \beta_1 k e_1 . \tag{17}$$

Here, the expressions for the f_0, g_0; f_1, g_1; F_0, G_0; and F_1, G_1 can be found from Eqs. (7), (9), (13), and (15), respectively.

The average phase velocity $\langle \dot{\varphi}_1 \rangle$ is equal to

$$\langle \dot{\varphi}_1 \rangle = \langle \dot{f}_0 \rangle + k \langle \dot{f}_1 \rangle + e_1 \langle \dot{F}_0 \rangle + e_1 k \langle \dot{F}_1 \rangle . \tag{18}$$

Using the standard Fourier transformation, we can find from Eq. (7)

$$\begin{aligned} \dot{f}_0 &= \omega_0 \left\{ 1 + \sum_{n=1}^{\infty} 2\bar{k}^n \cos\left[n \left[\omega_0 t + \frac{\pi}{2} + \theta_1 \right] \right] \right\} , \\ \dot{g}_0 &= \omega_0 \left\{ 1 + \sum_{n=1}^{\infty} 2\bar{k}^n \cos\left[n \left[\omega_0 t + \frac{\pi}{2} + \theta_2 \right] \right] \right\} , \end{aligned} \tag{19}$$

where $\bar{k} = e_0 - \omega_0$. Integrating Eq. (19), we get

$$f_0(t)=\omega_0\left\{t+\sum_{n=1}^{\infty}\frac{2\bar{k}^n}{n\omega_0}\sin\left[n\left[\omega_0 t+\frac{\pi}{2}+\theta_1\right]\right]+c_1\right\},$$

$$g_0(t)=\omega_0\left\{t+\sum_{n=1}^{\infty}\frac{2\bar{k}^n}{n\omega_0}\sin\left[n\left[\omega_0 t+\frac{\pi}{2}+\theta_2\right]\right]+c_2\right\}, \tag{20}$$

where

$$c_1=-\sum_{n=1}^{\infty}\frac{2\bar{k}^n}{n\omega_0}\sin\left[n\left[\frac{\pi}{2}+\theta_1\right]\right],$$

$$c_2=-\sum_{n=1}^{\infty}\frac{2\bar{k}^n}{n\omega_0}\sin\left[n\left[\frac{\pi}{2}+\theta_2\right]\right]. \tag{21}$$

Therefore, from Eqs. (20), we can find

$$\langle \dot{f}_0\rangle=\frac{1}{T}[f_0(T)-f_0(0)]=\omega_0 . \tag{22}$$

Here T is the intrinsic period of the system, $T=2\pi/\omega_0$. Using Eq. (20) and completing the integrals in Eq. (9), we can also obtain

$$\langle \dot{f}_1\rangle=\frac{1}{T}[f_1(T)-f_1(0)]$$
$$=\frac{e_0}{\omega_0^2}[\alpha_1 e_0+\omega_0 e_0(c_2-c_1)-\bar{k}\sin(\theta_2-\theta_1)] . \tag{23}$$

This term represents a small correction to the phase velocity of the coupled system due to coupling between two oscillators. This correction comes mainly from the difference between two initial phases θ_1 and θ_2 of the two oscillators. As will be seen in the following, it will have no effect on the harmonic or subharmonic steps.

In Eq. (13), using the equality

$$\frac{1}{e_0+\sin f_0}=\frac{1}{\dot{f}_0}=\frac{e_0+\sin(\omega_0 t+\theta_1)}{e_0^2-1}, \tag{24}$$

it is not difficult to demonstrate that the $F_0(t)$ term contributes only to the main step when $\omega=\omega_0$, and has no contribution at all to the subharmonic steps we are interested in when $\omega=n\omega_0$.

The term $F_1(t)$ is the most important and the only one having a contribution to the subharmonic steps. Substituting the expressions for f_1, F_0, and $G_0(t)$ into Eq. (15), and after making a very lengthy and tedious integral manipulation for the situation $\omega=n\omega_0$ $(n>1)$, we can find an expression for $F_1(t)$ and correspondingly get $\langle \dot{F}_1\rangle$. In the calculation, in order to avoid the secular term in $\langle \dot{F}_1\rangle$, following the rules in the singular perturbation method, the parameter b_0 should be set equal to zero. Because the formula of $\langle \dot{F}_1\rangle$ is too long to be written in this paper, and we are only interested in the coupling effect on the interference and the subharmonic steps, here we have written down its expression with only terms up to the zero order in $\bar{k}$,

$$\langle \dot{F}_1\rangle\approx\frac{e_0}{\omega_0^3}\left[e_0\omega_0\beta_1-\frac{e_0}{(n^2-1)\omega_0}(\cos\theta_2-\cos\theta_1)\right] . \tag{25}$$

Finally, from Eqs. (22), (23), and (25) and in the zero order of $\bar{k}$, we obtain

$$\langle \dot{\varphi}_1\rangle\sim\langle \dot{f}_0\rangle+e_1 k\langle \dot{F}_1\rangle . \tag{26}$$

At the same time, the external dc field equals

$$e=e_0+\beta_1 k e_1+\cdots . \tag{27}$$

Now we see from Eq. (25) that for the same values of β_1, the phases θ_1 and θ_2 of the two coupled oscillators can be adjusted to make the value in the square brackets of Eq. (25) become zero. If so, it means that when the external dc force e is varied about the e_0 in the scale of $\beta_1 k e_1$, the $\langle \dot{F}_1\rangle$ remains unchanged $(=0)$, and so $\langle \dot{\varphi}_1\rangle$ is not changed and still equals ω_0. Obviously, it means a subharmonic step satisfying $\omega=n\omega_0$ will appear in the dc response of the coupled system to an external force.

The step width can also be found from Eq. (25). When $\cos\theta_2-\cos\theta_1=\pm 2$, the β_1 takes its maximum and minimum values, respectively. So, $\delta\beta_1=\beta_{1\max}-\beta_{1\min}=4/[(n^2-1)\omega_0^2]$. The step width for $\omega=n\omega_0$ is given by

$$\Delta_{1/n}\approx\frac{4ke_1}{(n^2-1)\omega_0^2}, \tag{28}$$

which is proportional to k and e_1 in our approximation, and when $k\to 0$, i.e., going back to the overdamped single oscillator model, the width will approach zero, too.

From Eq. (28), we can see the following.

(1) The bigger the value of n, the narrower the width of the corresponding subharmonic step.

(2) The width Δ is proportional to the amplitude e_1 of the vibrating applied force. We should emphasize that this behavior is caused by our perturbation expansion to only the first order of the e_1. In general, the width Δ may have a more complicated dependence on e_1 [11].

(3) When $k\to 0$, $\Delta_{1/n}\to 0$, and our model reduces to the ordinary overdamped oscillator model.

We had investigated the single underdamped oscillator model with a small inertial term $\beta_m \ll 1$ [11] and found that

$$\Delta_{1/2}\approx\beta_m\frac{e_1\omega_0\bar{k}}{e_0} . \tag{29}$$

Comparing Eqs. (29) and (28), we may conjecture that the coupling term between two overdamped oscillators plays an effective inertial role and the

$$\beta_m\approx\tfrac{4}{3}k\frac{e_0}{\omega_0^3\bar{k}} \tag{30}$$

$$\approx\frac{8k}{3\omega_0}\quad(\text{when } e_0\gg 1) . \tag{31}$$

This result is qualitatively consistent with some numerical calculations [3] based upon the deformation model in the CDW system although they used a train of field pulses (from zero to above threshold) rather than using sinusoidal driving. Strogatz *et al.* [8] also found the inertialike effects (hysteresis, switching, etc.) in a model of coupled overdamped oscillators with a periodic coupling rather than a quadratic coupling used in our model. Because an ac driving field was not included in their calculation they did not investigate the combined effect of the dc and ac driving fields and, of course, did not find the subharmonic steps either. Both of their results and my work demonstrate the importance of the coupling term in coupled overdamped oscillators, i.e., the coupling can produce the inertialike effects in a whole region from a pure dc dynamical response to the interference effects between applied ac and dc fields. There are also differences between both results. They claimed the crucial dependence of their switching and hysteresis on the periodic coupling in their model, and said neither switching nor hysteresis was predicted for quadratic coupling used in our model. However, in our paper we have found the subharmonic steps in coupled overdamped oscillators with only quadratic coupling, which is another kind of inertial-like effect in the sliding state. After comparing with the result obtained in the single-underdamped-oscillator model [11], we think that the coupling term may produce an effective inertia mass in the dynamical response. If so, the system will be able also to show the hysteresis near the threshold field due to the existence of the effective inertia, as demonstrated numerically in Ref. [11]. Even though one does not believe in the existence of the effective inertia, it is still natural to infer physically the existence of the hysteresis due to the coupling term. This is because when the dc external field e is increased slowly past the threshold field e_T, the induced phase velocity $\langle \dot{\varphi}_i \rangle$ jumps discontinuously up to a finite value from zero, and the oscillator (or particle) will store up elastic energy due to the coupling term during this depinning process. Then it increases as e increases. However, when e is then decreased, the $\langle \dot{\varphi}_i \rangle$ also decreases and then drops discontinuously to zero at the separate pinning threshold value e'_T because it is now in a sliding-motion state and has a stored elastic energy, which is different from the initial state [when e increases from zero to above e_T). I think the stored elastic energy perhaps plays a role here similar to the inertial energy when the oscillator is an underdamped one with an inertia mass. Of course, the above conjecture or inference should be proved by a numerical simulation. This is an interesting problem and will be done in another paper.

Why can the coupling term between two coupled overdamped oscillators produce inertial effects in the sliding motion state? What is its physical reason? These questions should be given more investigation. Here, we will give a possible explanation. It is the coherence between different oscillators that causes the appearance of the inertial effects. In the pinned state, there are a lot of quasistationary equilibrium states for the coupled system and correlation between different parts in this coupled system is rather small. The random pinning fields make each oscillator have its independent φ_i. However, after the external field becomes larger than the e_T, the coupled oscillators go into a sliding motion state. The correlation effect becomes larger. It is just coupling term that makes each oscillator bind together, move coherently, and have an average common phase velocity $\langle \dot{\varphi}_i \rangle$. So, originally uncorrelated oscillators in the coupled system bind together to become a single moving rigid body, and most probably, its inertia cannot be again considered as negligible and perhaps even has a small value. Anyway, this coherence in the sliding state may be a possible physical reason to produce the inertial effect from the coupling of overdamped oscillators.

In conclusion, we have demonstrated that the interaction existing in a system with many degrees of freedom has a fundamental effect on its dynamical response to an external force, and the subharmonic steps can be only caused by the coupling terms under the approximation $e_1 < k \ll 1$. There will be no subharmonic steps at all without the coupling. This phenomenon may indicate the existence of an intrinsic relation between the "inertial" response of the nonlinear coupled system and the coupling between different degrees of freedom. This conclusion may be used for many practical physical systems, e.g., the CDW system, Josephson-junction arrays, etc.

I would like to thank Professor Lu Yu for helpful discussions. This work is supported by the State Education Commission.

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